Observations by Brian Woods on Comments by Professor Oskar Goecke on the Smoothing Formula in Colm Fagan's Entry for IFOA's Frank Redington Pension Prize: "A New Approach to Auto-Enrolled Pensions"

Brian Woods' note (below) summarises the key results and key messages from remarks by Professor Oskar Goecke on the smoothing formula in Colm Fagan's essay for the IFOA. Professor Goecke's remarks are attached as an appendix to Brian's note.

The smoothing formula in Colm Fagan's paper is as follows:

$$SV_t = CF_t + p(MV_t - CF_t) + (1-p)SV_{t-1}(1+i_{t-1})$$

SV and *MV* refer to smoothed value and market value of the total fund respectively, *t* is the period (e.g., months) of the application of the smoothing formula and *p* is the smoothing parameter chosen for the formula e.g., 1% for monthly.

Effect on reserve/ cover ratio

To simplify matters, we assume that $CF_t = 0$, i.e., incoming and outgoing cashflows cancel each other. Then the *Smoothing Formula* simplifies to:

$$SV_t = pMV_t + (1-p)SV_{t-1}(1+i_{t-1}) \qquad \textbf{(Eq 1)}$$

Some definitions to simplify notation:

$q_t := \frac{SV_t}{MV_t}$, the "inverse cover ratio"

$1 + i_t^{(P)} := \frac{MV_t - CF_t}{MV_{t-1}} = \frac{MV_t}{MV_{t-1}}$ (performance of the underlying portfolio observed at time *t*)[1]

Then we can rewrite (Eq 1): $q_t = p + (1-p)\frac{1+i_{t-1}}{1+i_t^{(P)}}q_{t-1}$ **(Eq 2)**

We define a simple stochastic model for $\frac{1+i_t^{(P)}}{1+i_{t-1}}$, namely

$\frac{1+i_t^{(P)}}{1+i_{t-1}} = exp(\delta + \sigma Z_t)$ with constant δ and σ and a standard normally distributed Z_t .

δ is the chosen "convergence" parameter of the smoothing formula and

σ is the assumed volatility for the log return of the fund's investments.

Remark: This simple model implicitly assumes $1 + i_t^{(P)} = exp(\mu_{t-1} + \delta_0 + \sigma Z_t)$ and that $1 + i_{t-1} = exp(\mu_{t-1} + \delta_0 - \delta)$ where μ_{t-1} is the *log-interest rate* of a riskless investment for [*t*-1, *t*] observable at time *t*-1, δ_0 is the equity risk premium. Furthermore, we assume that (Z_t) are stochastically independent. If $\delta = 0$ then i_{t-1} would be the expected total return on the fund's assets.

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[1] supposing that cashflow CF_t occurs at time *t*

Finally, we set $\xi := -ln(1-p)$ and $\delta' = \delta + \xi$.[2] Then we get the following recursion formula for q_t:

$$q_t = p + exp(-\xi - \delta - \sigma Z_t)\, q_{t-1} = p + exp(-\delta' - \sigma Z_t)\, q_{t-1} \qquad \textbf{(Eq 3)}$$

Remark:

$\mathbb{E}(q_t) = q_{t-1} \quad \Leftrightarrow \quad q_{t-1} = \frac{p}{1-(1-p)\,exp(\frac{1}{2}\sigma^2-\delta)}$.

Thus if $\mathbb{E}(q_t)$ does converge to some finite number q_∞ then $q_\infty = \frac{p}{1-(1-p)\,exp(\frac{1}{2}\sigma^2-\delta)}$.

We consider the question as to whether the system is stable in the long run:

The algebra in the appendix leads to the following results.

If $\frac{1}{2}\sigma^2 - \delta' = 0$ then $\mathbb{E}(q_k) = kp + q_0$ otherwise

$$\mathbb{E}(q_k) = p\ \frac{1-exp\left(k(\frac{1}{2}\sigma^2-\delta')\right)}{1-exp(\frac{1}{2}\sigma^2-\delta')} + exp\left(k(\frac{1}{2}\sigma^2 - \delta')\right) q_0$$

Thus if $\frac{1}{2}\sigma^2 - \delta' \geq 0$ then $\lim_{k\to\infty} \mathbb{E}(q_k) = +\infty$, and

if $\frac{1}{2}\sigma^2 - \delta' < 0$ then $\lim_{k\to\infty} \mathbb{E}(q_k) = \frac{p}{1-exp(\frac{1}{2}\sigma^2-\delta')} = \frac{p}{1-(1-p)\,exp(\frac{1}{2}\sigma^2-\delta)}$

Similarly, I think it can be shown that $\lim_{k\to\infty} Var(q_k)$ is finite if $\sigma^2 - \delta' < 0$, though Professor Goecke did not develop algebra for the variance.

The key message for me is:

> It would seem that we would wish the inverse cover ratio to converge to a value less than or equal to 1 viz. that $\frac{1-exp(\frac{1}{2}\sigma^2-\delta)(1-p)}{p} \geq 1$. This is equivalent to $exp(\frac{1}{2}\sigma^2 - \delta)$<=1 i.e. $\sigma^2 \leq 2\delta$. For example, we see that if we choose δ to be 0, that is if we chose our convergence parameter so that $i_t = \mathbb{E}(i_t^{(P)})$ then the inverse cover ratio will not converge to less than or equal to 1.

[2] We may assume that $p < 1$.

Observations from the stochastic model that we used

We ran 2,000 simulations over 60 years in quarterly periods. The underlying stochastic model was a simplified Wilkie with some auto-regression in the dividend yield. We also allowed for modelled cash flow so we can see that the Remarks discussed above addressed a more simplified scenario. None-theless it prompted me to look at the convergence characteristics of the simulations.

The underlying parameters in the simulations were as follow, using the above terminology.

t: time in quarters

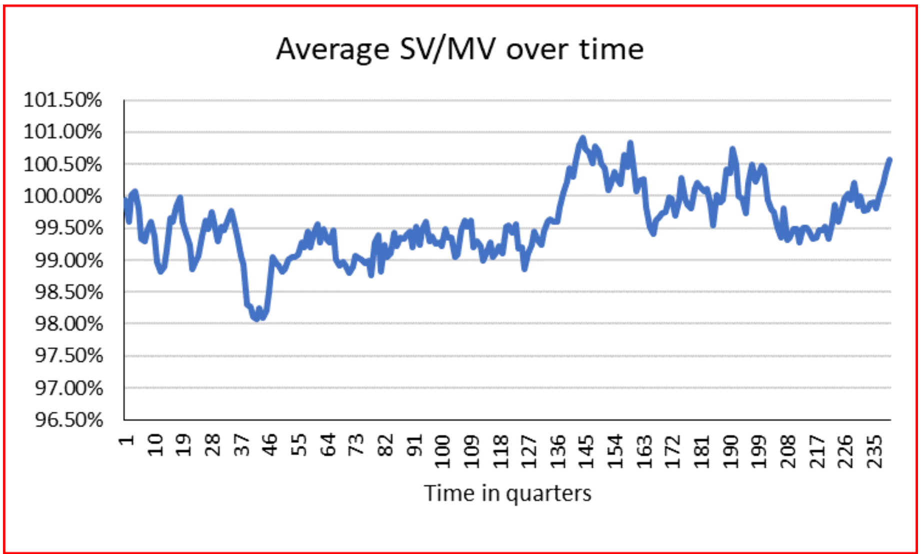
δ_0: 6.2% p.a.

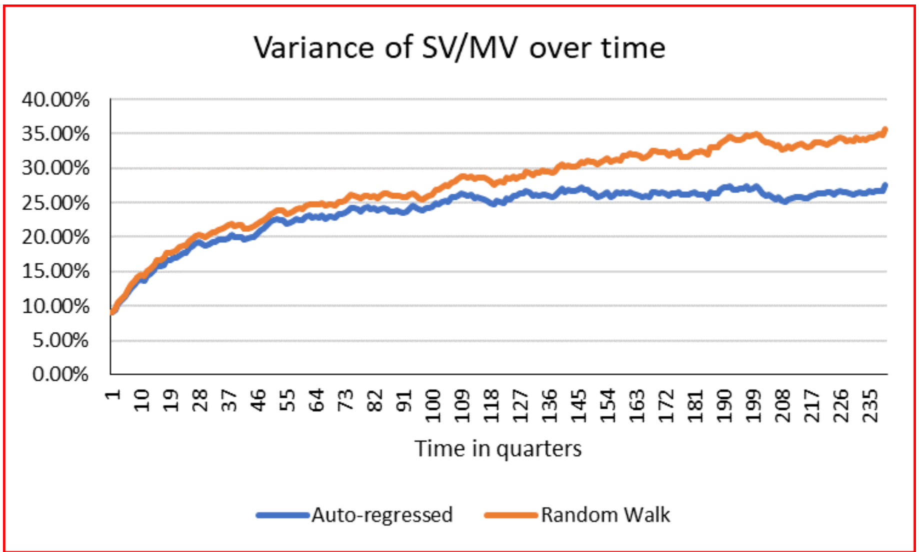
σ: 16.3% p.a.

p: 4.5% per quarter

i_t: 4% p.a => $\delta \cong 2\%$ p.a.

The below figures show the development of average and variance of SV/MV through the simulations.





Appendix: Remarks by Professor Oskar Goecke on
Colm Fagan: A New Approach to Auto-Enrolled Pensions (Oct. 2022)

Institute and Faculty of Actuaries

$$SV_t = CF_t + p(MV_t - CF_t) + (1-p)SV_{t-1}(1+i_{t-1}) \qquad (\textit{Smoothing Formula}, \text{ p. 8})$$

Effect on reserve/ cover ratio

To start with, we assume that $CF_t = 0$, i.e. the incoming and outgoing cashflow cancel each other. We could say that the pension system is in a *steady state*. Then the *Smoothing Formula* simplifies to:

$$SV_t = pMV_t + (1-p)SV_{t-1}(1+i_{t-1}) \qquad \textbf{(Eq 1)}$$

Some definitions to simplify notation

$q_t := \frac{SV_t}{MV_t}$ (*inverse of the cover ratio*),

$1 + i_t^{(P)} := \frac{MV_t - CF_t}{MV_{t-1}} = \frac{MV_t}{MV_{t-1}}$ (performance of the underly portfolio observed at time t)[3]

Then we can rewrite (Eq 1):

$$q_t = p + (1-p)\frac{1+i_{t-1}}{1+i_t^{(P)}} q_{t-1} \qquad \textbf{(Eq 2)}$$

We define a simple stochastic model for $\frac{1+i_t^{(P)}}{1+i_{t-1}}$, namely

$\frac{1+i_t^{(P)}}{1+i_{t-1}} = exp(\delta + \sigma Z_t)$ with constant δ and σ and a standard normally distributed Z_t .

Remark: This simple model implicitly assumes that $1 + i_{t-1} = exp(\mu_{t-1} + \delta_1)$ is and $1 + i_t^{(P)} = exp(\mu_{t-1} + \delta_0 + \sigma Z_t)$, where μ_{t-1} is the *log-interest rate* of a riskless investment for [t-1, t] observable at time t-1, δ_0 is the equity risk premium and δ_1 is the share in the equity risk premium. Furthermore, we assume that (Z_t) are *stochastically independent*. δ can be interpreted as the *risk premium* we get in return for accepting volatility. We define $\delta := \delta_0 - \delta_1$, then $\delta = 0$ means that i_{t-1} is the *expected total return on the plan assets*.

Finally, we set $\xi := -ln(1-p)$ and $\delta' := \delta + \xi$.[4] Then we get the following recursion formula for q_t:

$$q_t = p + exp(-\xi - \delta - \sigma Z_t)\, q_{t-1} = p + exp(-\delta' - \sigma Z_t)\, q_{t-1} \qquad \textbf{(Eq 3)}$$

Remarks:

[3] supposing that the cashflow CF_t occurs at time t
[4] We may assume that $p < 1$.

1. $\mathbb{E}(q_t) = q_{t-1} \quad \Leftrightarrow \quad q_{t-1} = \frac{p}{1-(1-p)\,exp(\frac{1}{2}\sigma^2-\delta)}$.

2. Suppose that we wanted to avoid underfunding, i.e. $q_t > 1$. Then by (Eq 3) we can calculate $\mathbb{P}(q_t > 1) = 1 - \Phi\left(\frac{1}{\sigma}\left(ln\left(\frac{1-p}{q_{t-1}}\right) + \delta'\right)\right)$. Given a "*safety level*" $\alpha > 0$, we get

$$\mathbb{P}(q_t > 1) \le \alpha \quad \Leftrightarrow \quad ln\left(\frac{1-p}{q_{t-1}}\right) + \delta' \ge \sigma u_{1-\alpha} \Leftrightarrow \quad \delta + ln\left(\frac{1}{q_{t-1}}\right) \ge \sigma u_{1-\alpha}$$

   E.g., for $\sigma = 15\%$, $\delta = 0$ and $\alpha = 5\%$ we get

$$\mathbb{P}(q_t > 1) \le 5\% \quad \Leftrightarrow \quad ln\left(\frac{1}{q_{t-1}}\right) \ge 0.2467 \Leftrightarrow \quad \frac{1}{q_{t-1}} \ge 127.98\%.$$

Now we come to the question whether the system is stable on the long run:

For $t_0 = 0$ we get by induction:

$$q_k = p + p\sum_{j=1}^{k-1} exp\left(-j\delta' - \sigma(Z_{k-j+1}+\ldots+Z_k)\right) + exp(-k\delta' - \sigma(Z_1+\ldots+Z_k))\,q_0$$

(Eq 4)

By setting $W_j := -(Z_{k-j+1}+\ldots+Z_k)$ for $j = 1,\ldots, k\text{-}1$ this simplifies to

$$q_k = p + p\sum_{j=1}^{k-1} exp(-j\delta' + \sigma W_j) + exp(-k\delta' - \sigma W_k)\,q_0 \qquad \textbf{(Eq 5)}$$

Since $\mathbb{E}\left(exp(-j\delta' + \sigma W_j)\right) = exp\left(j(\frac{1}{2}\sigma^2 - \delta')\right)$:

$$\mathbb{E}(q_k) = p + p\sum_{j=1}^{k-1} exp\left(j(\frac{1}{2}\sigma^2 - \delta')\right) + exp\left(k(\frac{1}{2}\sigma^2 - \delta')\right) q_0 \qquad \textbf{(Eq 6)}$$

If $\frac{1}{2}\sigma^2 - \delta' = 0$ then $\mathbb{E}(q_k) = kp + q_0$ otherwise

$$\mathbb{E}(q_k) = p\ \frac{1-exp\left(k(\frac{1}{2}\sigma^2-\delta')\right)}{1-exp(\frac{1}{2}\sigma^2-\delta')} + exp\left(k(\frac{1}{2}\sigma^2 - \delta')\right) q_0$$

If $\frac{1}{2}\sigma^2 - \delta' \ge 0$ then $\lim_{k\to\infty}\mathbb{E}(q_k) = +\infty$, and

if $\frac{1}{2}\sigma^2 - \delta' < 0$ then $\lim_{k\to\infty}\mathbb{E}(q_k) = \frac{p}{1-exp(\frac{1}{2}\sigma^2-\delta')} = \frac{p}{1-(1-p)\,exp(\frac{1}{2}\sigma^2-\delta)}$

Remarks:

- $\lim_{\substack{p\to 1\\ p<1}} \frac{p}{1-exp(\frac{1}{2}\sigma^2-\delta')} = 1$; this is consistent with the trivial case $p = 1$.
- $\frac{1}{2}\sigma^2 - \delta' < 0 \Leftrightarrow \frac{1}{2}\sigma^2 < \delta - ln(1-p)$; this is true for reasonable σ- and δ-values.
- The inverse cover ratio should converge to a value less equal 1 or

  $\frac{1-exp(\frac{1}{2}\sigma^2-\delta')}{p} \ge 1$. This is equivalent to $exp(\frac{1}{2}\sigma^2 - \delta) \le 1$.

We now skip the assumption $CF_t = 0$, i.e. we allow for positive or negative cashflows.

We consider the *inverse of the cover ratio* just a second before the net cashflow is accounted for

$$q_t := \frac{SV_t - CF_t}{MV_t - CF_t}$$

and we introduce a new variable, namely

$$\tau_t := ln\left(\frac{\frac{SV_t}{MV_t}}{\frac{SV_t - CF_t}{MV_t - CF_t}}\right) = ln\left(\frac{SV_t}{MV_t}\right) - ln\left(\frac{SV_t - CF_t}{MV_t - CF_t}\right).$$

τ_t is a measure for the impact of the cashflow CF_t on the inverse cover ratio. Note that in the case that $\frac{SV_t - CF_t}{MV_t - CF_t} < 1$ (this should be the normal case!), a positive cash flow results in *lower* cover ratio (i.e. $\tau_t > 0$).

The we can rewrite (**Eq 2**) and (**Eq 3**)

$$q_t = p + (1 - p)\frac{1 + i_{t-1}}{1 + i_t^{(P)}} exp(\tau_{t-1}) q_{t-1} \qquad \textbf{(Eq 7)}$$

$$q_t = p + exp(\tau_{t-1} - \xi - \delta - \sigma Z_t)\, q_{t-1} = p + exp(\tau_{t-1} - \delta' - \sigma Z_t)\, q_{t-1} \qquad \textbf{(Eq 8)}$$

Literature

BaFin (2017): Bundesanstalt für Finanzdienstleistungsaufsicht, Merkblatt- Hinweise für die Zulassung von Pensionsfonds, 11.12.2017, https://www.bafin.de/SharedDocs/Veroeffentlichungen/DE/Merkblatt/VA/mb_171211_pensionsfonds_va.html